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TECHNICAL NOTES

The transient response of finned crossflow heat exchangers

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1. INTRODUCTION

INCREASING interest is focusing on the transient behaviour of heat exchangers due to advances in process control. To allow for full automation in a process plant, the dynamic behaviour of each component must be known. In general, the dynamic behaviour is gained by mathematical modelling since experimental determination is seldom possible. With reference to heat exchangers, such a model describes the time-dependent outlet fluid temperatures due to the inlet fluid temperature or mass flow perturbations. It is of advantage to have these models in an analytical formulation, since numerical models involve extensive programming.

Dealing with crossflow heat exchangers, there will be at least two spacial coordinates besides the time variable, thus complicated formulae result even for the simplest type, the plain plate type crossflow heat exchanger[1]. Because of these mathematical problems, the fins encountered in most crossflow heat exchangers have been either neglected [2, 3], or they have been modelled as a system with lumped parameters [4]. As was shown by Müller [5], the transient heat conduction in the fin has a considerable influence on the dynamics of the heat exchanger. Analytical models which include this effect, thus exhibiting reasonable accuracy, seem possible only with the basic element method first used by Müller. According to this method the heat exchanger is divided into small, geometric simple parts, the 'basic elements' are shown in Fig. 1. A mathematical model for such an element describing the time-dependent fluid temperatures is derived and then these elements are recombined to give the dynamics of the entire heat exchanger (Fig. 1). Since this piecewise recombination is shown to be quite simple, a whole variety of crossflow heat exchangers varying in size and configuration may thus be modelled using the same basic element.

This method was used by Ober [6] and Schmachtenberg [7] to establish models valid for special heat exchangers used in air-conditioning systems. Their basic elements were fairly simple and calculated fluid exit temperatures due to temperature perturbation only. Further literature is cited in ref. [8]. The aim of this article is an advanced analytical model describing the transient response of crossflow heat exchangers initiated by arbitrary time-varying inlet temperatures or mass flow rates.

2. THE BASIC ELEMENT

The crossflow heat exchanger is divided into basic elements which allow for realistic modelling. A basic element is chosen to be a section of the core tube carrying exactly one square fin (Fig. 1). The transient behaviour of this basic element is governed by the energy conservation law formulated for the fluid in the tube (which is assumed to be a liquid indexed l), the tube wall (indexed t), the fin (indexed f) and the fluid on the outside, a gas with the index g. The resulting differential equations are simplified by the following assumptions.

(1) Heat conduction in the fin is allowed only in the y-direction (perpendicular to gas flow, see Fig. 2).
(2) The temperature profile of the gas in the direction of flow (x-coordinate) is approximated by a linear function.
(3) There is no temperature dependence of material properties.
(4) There is no energy transport in the gas except in the direction of flow.
(5) Liquid temperature and velocity are constant in the radial and circumferential direction.
(6) Energy transport in the liquid due to the deviation from plug flow is considered by means of a dispersion factor λ_{disp}, which is handled as an enlarged thermal conductance for the liquid.
(7) Liquid and gas velocity are constant throughout the basic element.

The first two assumptions, which simplify the equations significantly, are valid over a wide range of parameters. This has been shown by comparison with a more sophisticated model in ref. [8]. The time-dependent variables $\dot{m}$, θ and h are divided into a stationary and a transient value in order to avoid complex convolution integrals when performing the latter following Laplace transform:

$$\dot{m} = \bar{\dot{m}} + \tilde{\dot{m}}; \quad \theta = \bar{\theta} + \tilde{\theta}; \quad h = \bar{h} + \tilde{h}.$$

Because products of two transient values are neglected, only small deviations about the stationary values are allowed.† In addition the dependence of the heat transfer coefficient h on velocity is linearized as follows:

$$\frac{h}{\bar{h}} = \left(\frac{w}{\bar{w}}\right)^k = \left(\frac{\bar{w}+\tilde{w}}{\bar{w}}\right)^k \simeq 1 + k\frac{\tilde{w}}{\bar{w}}.$$

The resulting equations describing the temperatures in the basic element are given in dimensionless form by equations (1)–(5)

$$\frac{\partial \tilde{\theta}_{g,t}}{\partial \tau} + \frac{\partial \tilde{\theta}_{g,t}}{\partial x} + m_g \frac{\partial \tilde{\theta}_{g,t}}{\partial x} = St_{g,t}(\tilde{\theta}_t - \tilde{\theta}_{g,t}) + St_{g,t}\, k_g m_g (\bar{\theta}_t - \bar{\theta}_{g,t})$$

$$\tau = 0: \tilde{\theta}_{g,t} = 0 \quad x = 0: \tilde{\theta}_{g,t} = \tilde{\theta}_{g,in} \quad \text{gas } 0 \leqslant y < h_1 \quad (1)$$

† This only concerns transients in mass flow.

NOMENCLATURE

h	heat transfer coefficient [W m^{-2} K^{-1}]
G	transfer function
k	exponent describing the velocity dependence of h
$\mathscr{L}\{f\}$	Laplace-transformed function f
m	dimensionless mass flow rate (see Table 1)
$\dot{m}$	mass flow rate [kg s^{-1}]
M	Laplace-transformed mass flow rate m
s	complex Laplace variable
t_0	dwell time of the gas in a basic element, L/w_g [s]
w_l, w_g	fluid velocity [m s^{-1}]
z_t^*	graduation of the finned tube (length of the core tube of a basic element) [m].

Greek symbols

θ	dimensionless temperature
Θ	Laplace-transformed temperature θ
τ	dimensionless time, t/t_0
ω	dimensionless circular frequency, $\omega^* t_0$.

Subscripts

f	fin
g	gas, zone I in Fig. 2
g, t	gas, zone II in Fig. 2
l	liquid
t	tube.

Superscripts

S	synthesis transfer function
$\bar{}$	stationary value (working point)
$\tilde{}$	transient value.

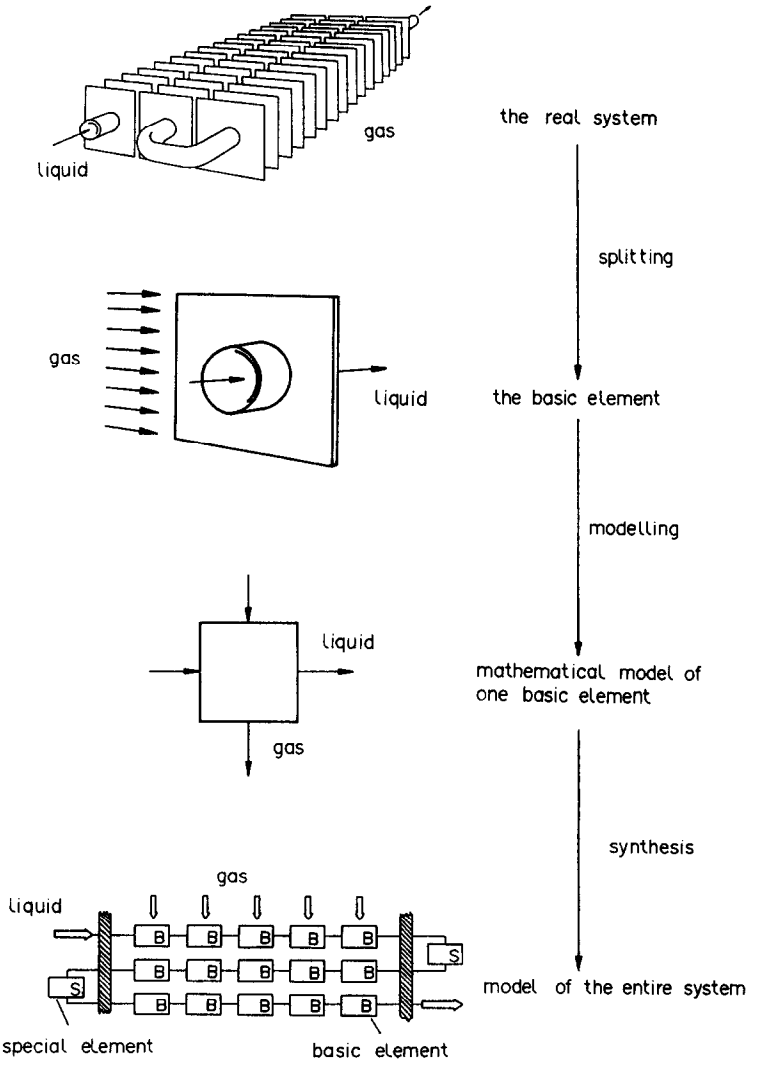

FIG. 1. Principle procedures in modelling a finned heat exchanger.

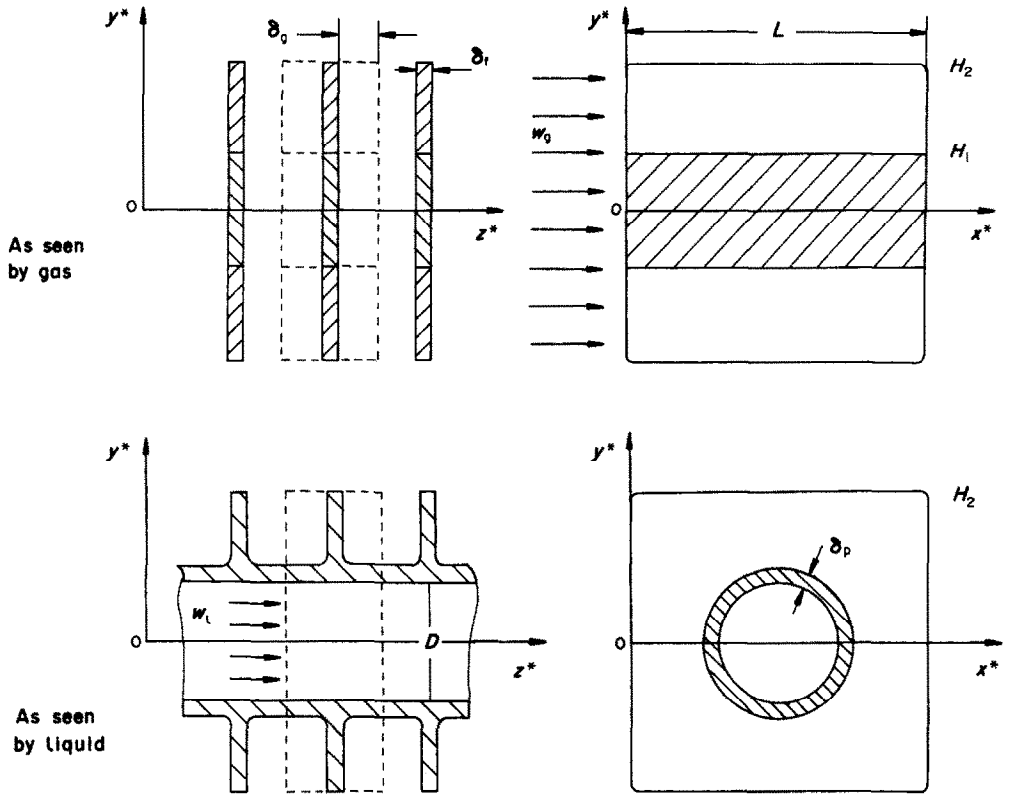

FIG. 2. Model of the basic element.

$$\frac{\partial\tilde{\theta}_g}{\partial\tau}+\frac{\partial\tilde{\theta}_g}{\partial x}+m_g\frac{\partial\tilde{\theta}_g}{\partial x}=St_g(\tilde{\theta}_f-\tilde{\theta}_g)+St_g k_g m_g(\tilde{\theta}_f-\tilde{\theta}_g)$$

$$\tau=0:\tilde{\theta}_g=0 \quad x=0:\tilde{\theta}_g=\tilde{\theta}_{g,in} \quad \text{gas } h_1\leqslant y<h_2 \tag{2}$$

$$\frac{\partial\tilde{\theta}_f}{\partial\tau}=Fo_f\frac{\partial^2\tilde{\theta}_f}{\partial y^2}+St_f(\tilde{\theta}_g-\tilde{\theta}_f)+St_f k_g m_g(\tilde{\theta}_g-\tilde{\theta}_f)$$

$$\tau=0:\tilde{\theta}_f=0 \quad y=h_2:\partial\tilde{\theta}_f/\partial y=0 \quad y=h_1:\tilde{\theta}_f=\tilde{\theta}_t \quad \text{fin} \tag{3}$$

$$\frac{\partial\tilde{\theta}_t}{\partial\tau}=St_{t,l}(\tilde{\theta}_l-\tilde{\theta}_t)+St_{t,l}k_l m_l(\tilde{\theta}_l-\tilde{\theta}_t)+St_{t,g}(\tilde{\theta}_{g,t}-\tilde{\theta}_t)$$
$$+St_{t,g}k_g m_g(\tilde{\theta}_{g,t}-\tilde{\theta}_t)+Fo_t\left.\frac{\partial\tilde{\theta}_f}{\partial y}\right|_{y=h_1}$$

$$\tau=0:\tilde{\theta}_t=0 \quad \text{tube} \tag{4}$$

$$\frac{\partial\tilde{\theta}_l}{\partial\tau}+\omega\frac{\partial\tilde{\theta}_l}{\partial z}+m_l w\frac{\partial\tilde{\theta}_l}{\partial z}=Fo_l\frac{\partial^2\tilde{\theta}_l}{\partial z^2}+St_l(\tilde{\theta}_t-\tilde{\theta}_l)$$
$$+St_l k_l m_l(\tilde{\theta}_t-\tilde{\theta}_l)$$

$$\tau=0:\tilde{\theta}_l=0 \quad z=0:\tilde{\theta}_l=\tilde{\theta}_{l,in} \quad z\to\infty:\tilde{\theta}_l \quad \text{finite liquid.} \tag{5}$$

The dimensionless variables used are given in Table 1, the Stanton numbers St and the Fourier numbers Fo are defined in Table 2. Formulating differential equations (1) and (2) for the gas, the three-dimensional flow of the gas around the core tube has been neglected. Because the heat transfer between the tube and the gas is essential in describing the outlet temperatures, it is taken into account by a special zone on the fin. The fin is divided into two regions (Fig. 2), whereby zone II in the middle of the fin is a substitute for the tube (as far as the heat transfer to the gas is concerned). The area of this zone is given by the cross-sectional area of the pipe to $2LH_1=\pi D^2/4$ (Fig. 2), so that the area and mass of the fin (zone I) are unaffected. Zone II is modelled as a system with lumped parameters. Heat transfer from zone II is governed by a heat transfer coefficient valid for tubes in perpendicular crossflow, while for zone I the heat transfer coefficient for a flat plate is chosen.

The stationary values for the temperature difference $(\tilde{\theta}_i-\tilde{\theta}_j)$ shown in equations (1)–(5) depend on the independent variables x, y and z. Thus before the equations can be solved these temperature differences have to be

Table 1. Dimensionless variables

$$\theta=\frac{\theta^*}{1\,\mathrm{K}}$$

$$\tau=\frac{t}{t_0} \quad t_0=\frac{L}{w_g} \quad \text{dwell time of the gas}$$

$$x=\frac{x^*}{L};\quad y=\frac{y^*}{L};\quad z=\frac{z^*}{L} \quad L=\text{length of one fin}$$

$$m=\frac{\tilde{m}}{\bar{m}} \quad \bar{m}=\text{mass flow rate in the working point}$$

$$h_1=\frac{H_1}{L};\quad h_2=\frac{H_2}{L};\quad z_t=\frac{z_t^*}{L};\quad w=\frac{\bar{w}_l}{\bar{w}_g}$$

Table 2. Dimensionless numbers

Stanton numbers

$$St_g = \frac{\bar{h}_g}{\rho_g c_{p,g} \bar{w}_g} \frac{L}{\delta_g}; \quad St_{g,t} = \frac{\bar{h}_{g,t}}{\rho_g c_{p,g} \bar{w}_g} \frac{\pi(D+2\delta_t)}{2H_1}$$

$$St_f = \frac{\bar{h}_g}{\rho_f c_f \bar{w}_g} \frac{2L}{\delta_f}; \quad St_{t,g} = \frac{\bar{h}_{g,t}}{\rho_t c_t \bar{w}_g} \frac{2L\delta_g(D+2\delta_t)}{z_t \delta_t (D+\delta_t)}$$

$$St_l = \frac{\bar{h}_l}{\rho_l c_l \bar{w}_g} \frac{4L}{D}; \quad St_{t,l} = \frac{\bar{\alpha}_l}{\rho_t c_t \bar{w}_g} \frac{LD}{\delta_t (D+\delta_t)}$$

Fourier numbers

$$Fo_f = \frac{\lambda_f}{\rho_f c_f} \frac{t_0}{L^2}; \quad Fo_l = \frac{\lambda_l^{\text{Dispers}}}{\rho_l c_l} \frac{t_0}{L^2}$$

$$Fo_t = \frac{\lambda_f}{\rho_t c_t} \frac{t_0}{Lz_t^*} \frac{2\delta_f L}{\delta_t \pi(D+\delta_t)} = \frac{\lambda_f}{\rho_t c_t} \frac{2\delta_f L}{\bar{w}_g z_t^* \delta_t \pi(D+\delta_t)}$$

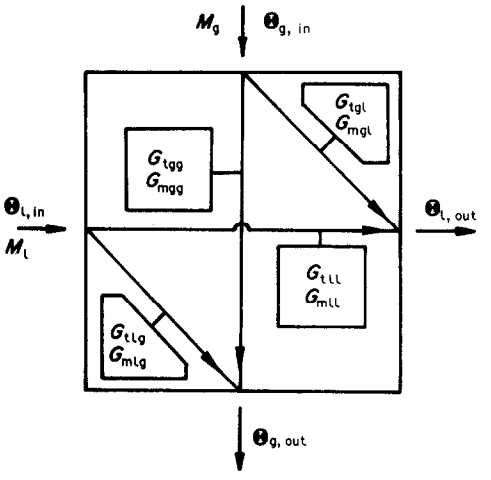

FIG. 3. The signal transfer in a basic element given by transfer functions.

replaced by the solutions of equations (1)–(5) formulated for the stationary case ($\partial\theta/\partial\tau = 0$).

Equations (1)–(5) are now Laplace transformed, whereby the temperatures are represented by $\mathscr{L}\{\bar{\vartheta}(t)\} = \Theta(s)$ and the mass flow rates by $\mathscr{L}\{\dot{m}(t)\} = M(s)$. Solving the system of differential equations in the Laplace domain leads to the transformed liquid and gas temperature at the outlet of the basic element, equations (6) and (7)

$$\Theta_{l,out} = G_{tll}\Theta_{l,in} + G_{tgl}\Theta_{g,in} + G_{mll}M_l + G_{mgl}M_g \quad (6)$$

$$\Theta_{g,out} = G_{tlg}\Theta_{l,in} + G_{tgg}\Theta_{g,in} + G_{mlg}M_l + G_{mgg}M_g. \quad (7)$$

Functions G accompanied with the inlet temperatures and mass flow rates are the transfer functions of the basic element. They are defined as the ratio of a Laplace transformed outlet value to a transformed inlet value and describe the signal transfer within one basic element as shown in Fig. 3. For example $G_{tlg} = \Theta_{g,out}/\Theta_{l,in}$ describes the behaviour of the gas temperature at the exit of the basic element dependent of the liquid inlet temperature $\Theta_{l,in}$. Transfer functions G are given in the Appendix.

3. SYNTHESIS OF BASIC ELEMENTS

Now that the dynamic behaviour of one basic element is known these elements have to be recombined to give the transients of an entire crossflow heat exchanger. For this task it is favorable to remain in the Laplace domain, since the linkage of transfer functions is simple. The series arrangement of transfer functions is represented by multiplication, the transfer function of a parallel arrangement is given by adding the transfer functions of each basic element. Thus, for example, the synthesis transfer function G_{tll}^S describing the liquid outlet temperature as a response to a change in liquid inlet temperature of a two row finned crossflow heat exchanger shown in Fig. 4 is given by

$$G_{tll}^S = \frac{\Theta_{l,out}}{\Theta_{l,in}} = G_u G_{tll}^{2M} + \sum_{i=1}^{M} G_{tlg} G_{tgl} G_{tll}^{2(i-1)} \quad (8)$$

where M is the number of basic elements in one row. The transfer function G_u takes into account the influence of the connection pipe between the first and the second row (see the Appendix). The first term in equation (8) describes the signal path within the liquid, the second term accounts for the signal transfer from the liquid to the gas and back to the liquid through the basic element behind. Because these signal paths are all possible simultaneously, they have to be added up in the overall transfer function (equation (8)). Another example, equation (9), shows the synthesis transfer function G_{mll}^S, which gives the changes in liquid outlet temperature due to a change in liquid mass flow rate (Fig. 4)

$$G_{mll}^S = \frac{\Theta_{l,out}}{M_l} = G_u \cdot G_{mll} \sum_{i=1}^{M} G_{tll}^{2M-i} + G_{mll} \sum_{i=1}^{M} G_{tll}^{M-i} + G_{mlg} G_{tgl} \sum_{i=1}^{M} G_{tll}^{M-i} + G_{mll} G_{tlg} G_{tgl} \sum_{l=1}^{M-1} \sum_{i=1}^{M-l} G_{tll}^{2(M-l)-i}. \quad (9)$$

The variational mass flow creates a change in temperature within each basic element. These temperature signals must be added to the incoming temperature signal for this element, so that complicated synthesis transfer functions result. Regarding the transfer functions concerned with mass flow variation, the stationary temperature difference $(\bar{\vartheta}_{l,in} - \bar{\vartheta}_{g,in})$ must be known for each basic element before the transient case can be calculated. This is simply done by the given transfer functions, since the stationary temperatures result when the Laplace parameter s is set to $s = (0, 0)$ [9]. The recurrence formulae for an arbitrary cross-parallel flow type of heat exchanger (as shown for example in Fig. 4) are given by equations (10) and (11)

$$\Theta_l^{n,m} = G_{tll}\Theta_l^{n,j-1} + G_{tgl}\Theta_g^{n-1,m} + G_{mll}M_l + G_{mgl}M_g \quad (10)$$

$$\Theta_g^{n,m} = G_{tlg}\Theta_l^{n,j-1} + G_{tgg}\Theta_g^{n-1,m} + G_{mlg}M_l + G_{mgg}M_g \quad (11)$$

where n counts the rows (liquid entry: $n = 1$) and m counts the basic elements (fins) in a row. Because of the deflection after each row one has to be able to distinguish between an odd and an even row index n: $j = m$ for odd n; $j = M - m + 3$ for even n. This simple 'element by element' recombination is possible because the inlet temperature functions of the first element are given for both liquid and gas. Thus the outlet temperatures can be calculated for this element preparing the computation of the second element and so forth. This is not successful for modelling a cross-*counterflow* heat exchanger (where the gas flow is opposite to the direction shown in Fig. 4), because in this case either the gas or liquid inlet temperature is unknown for *each* basic element. Thus a system of equations involving $2MN$ unknown variables must be solved to get an exact solution. For this type of heat exchanger, it is favorable to use approximate synthesis transfer functions. These approximate functions, equations (12), result when one transfer function of the basic element is

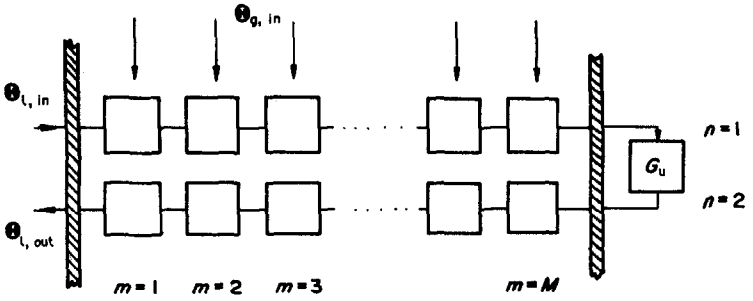

FIG. 4. The synthesis of a two row cross-parallel flow heat exchanger.

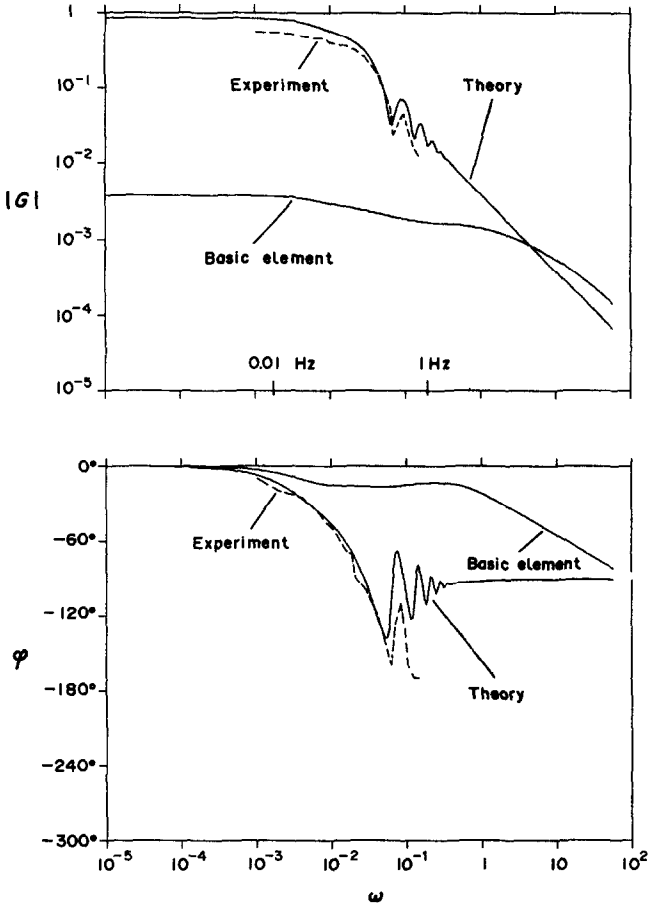

FIG. 5. Comparing experimental and theoretical data in a Bode diagram. The synthesis transfer functions G^S_{mll} and the same function of one basic element are shown.

neglected while recombining the elements. Predestinated for neglection is the transfer function G_{tgl}, responsible for the liquid exit temperature due to perturbation in the gas inlet temperature. There is virtually no effect in liquid temperature when the gas temperature varies, since the product of mass flow rate and heat capacity for the liquid, $\dot{m}_l \cdot c_l$, is huge compared with the value for the gas. Thus the following approximate synthesis transfer functions for cross-counter-flow type of heat exchangers are in good agreement with exact solutions

$$G^S_{tll} \simeq G^{NM}_{tll} G^{N-1}_u, \quad G^S_{tlg} \simeq \sum_{l=1}^{N} G^{l-1}_u G^{[M(l-1)+M/2]}_{tll} G^{l-1}_{tgg} G_{tlg}$$

$$G^S_{tgg} \simeq G^N_{tgg}, \quad G^S_{tgl} \simeq \sum_{l=1}^{N} \sum_{k=1}^{M} G^{l-1}_u G^{M(l-1)}_{tll} G^{l-1}_{tgg} G^{k-1}_{tll} G_{tgl}$$

$$G^S_{mll} \simeq \sum_{l=1}^{N} \sum_{k=1}^{M} G^{l-1}_u G^{Ml-k}_{tll} G_{mll}(\bar{\vartheta}^{N-l+1,M/2}_{l,g,in})$$

$$G^S_{mgl} \simeq \sum_{l=1}^{N} \sum_{k=1}^{M} G^{l-1}_u G^{Ml-k}_{tll} G_{mgl}(\bar{\vartheta}^{N-l+1,M/2}_{l,g,in})$$

$$G^{S}_{mlg} \simeq \sum_{l=1}^{N} G^{l-1}_{tgg} G_{mlg}(\bar{\theta}^{N-l+1,M/2}_{l,g,in}) + \sum_{k=1}^{M/2} G^{k}_{tll} G_{tlg} G_{mll}(\bar{\theta}^{l,M/4}_{l,g,in}) + \sum_{l=2}^{N} \sum_{k=1}^{M/2} G^{l-1}_{u} G^{M(l-1)+k}_{tll} G^{l-1}_{tgg} G_{tlg} G_{mll}(\bar{\theta}^{l,M/2}_{l,g,in})$$

$$G^{S}_{mgg} \simeq \sum_{l=1}^{N} G^{l-1}_{tgg} G_{mgg}(\bar{\theta}^{l,M/2}_{l,g,in}) + \sum_{k=1}^{M/2} G^{k}_{tll} G_{tlg} G_{mgl}(\bar{\theta}^{l,M/4}_{l,g,in}) + \sum_{l=2}^{N} \sum_{k=1}^{M/2} G^{l-1}_{u} G^{M(l-1)+k}_{tll} G^{l-1}_{tgg} G_{tlg} G_{mgl}(\bar{\theta}^{l,M/2}_{l,g,in}). \quad (12)$$

The synthesis transfer functions giving the gas exit temperature Θ_g refer to a position in the middle of the last row. The stationary temperature difference in the kth element of the ith row is denoted by $\bar{\theta}^{i,k}_{l,g,in}$. As an example the overall transfer function G^{S}_{mll} is shown in the Bode diagram (Fig. 5), valid for a cross-parallel flow heat exchanger consisting of one row with $M = 450$ basic elements. For this specific heat exchanger there exists experimental data from Sandbrink [4], which is included in the diagram (dashed line in Fig. 5). In addition the transfer function G_{mll} of one basic element is shown for comparison. Theory and experiment agree quite well. The interference phenomenon occurs at those frequencies, the reciprocal values of which are equal to the dwell time of liquid or gas.

4. THE RESULTS IN THE TIME DOMAIN

If the results discussed so far are not to be used in the process control it might be helpful to transfer them back into the time domain. The exit temperatures in the Laplace domain are given by

$$\Theta_{l,out} = G^{S}_{tll}\Theta_{l,in} + G^{S}_{tgl}\Theta_{g,in} + G^{S}_{mll}M_l + G^{S}_{mgl}M_g \quad (13)$$

$$\Theta_{g,out} = G^{S}_{tlg}\Theta_{l,in} + G^{S}_{tgg}\Theta_{g,in} + G^{S}_{mlg}M_l + G^{S}_{mgg}M_g. \quad (14)$$

A transformation back into the time domain is possible by means of an algorithm described by Chu *et al.* [10]. It is based on the equation

$$f(\tau) \simeq \frac{e^{c\tau^0}}{\tau^0}\left\{\frac{1}{2}F(c) + \mathscr{R}\sum_{k=1}^{N}(-1)^k F(c + ik\pi/\tau^0)\right\} \quad (15)$$

where F is the complex transfer function which is to be transferred back into the time domain, either $\Theta_{l,out}$ or $\Theta_{g,out}$ given by equation (13) or (14). τ^0 is a multiple of a time step $\Delta\tau$, by which the time domain function is assembled. c is a real number governing the accuracy of the algorithm. It should be chosen as $c \cdot \tau^0 = 5$ to give the best results.

Before evaluating equation (13) or (14) the input functions $\Theta_{l,in}, \ldots,$ etc. have to be specified as functions of s, for example the step function or the ramp function. If the input value is stationary, the transformed input function is zero.

5. CONCLUSIONS

A model to calculate the transient response of the liquid and gas outlet temperatures of a finned crossflow heat exchanger is described. The model allows for inlet temperature and mass flow rate perturbations, the results are given (analytically) as transfer functions or (numerically) in the time domain. Due to the basic element approach all different configurations and sizes of crossflow heat exchangers may be modelled. The agreement with experimental data is good, the analytical nature of the transfer functions allows convenient handling and discussion.

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APPENDIX: THE TRANSFER FUNCTIONS

The transfer functions resulting from the Laplace-transformed equations (1)–(5) are given as

$$G_{tgl} = \frac{1-e^{rz_t}}{\frac{s}{St_l}n + n - St_{t,l}} \times \left\{Fo_t\, a \tanh[a(h_2-h_1)]\frac{St_f}{Fo_f}\frac{2}{s+St_g+2}\frac{1}{a^2} + \frac{2St_{t,g}}{s+St_{g,t}+2}\right\}, \quad G_{tll} = e^{rz_t}$$

$$G_{mgl} = \frac{1-e^{rz_t}}{\frac{s}{St_l}n + n - St_{t,l}}\left\{Fo_t(a\tanh[a(h_2-h_1)] - a_0\tanh[a_0(h_2-h_1)])\frac{St_f}{Fo_f}\frac{2}{St_g+2}\frac{1}{a^2-a_0^2}\left[\frac{k_g(s+2)+St_g}{s+St_g+2}\right] + \frac{2St_{t,g}}{St_{g,t}+2}\left[\frac{k_g(s+2)+St_{g,t}}{s+St_{g,t}+2}\right]\right\}\frac{e^{r_0z_t}-1}{r_0z_t}\frac{St_{t,l}}{n_0}(\bar{\theta}_{l,in}-\bar{\theta}_{g,in})$$

$$G_{mll} = \left\{\frac{1-e^{rz_t}}{s/St_l + 1 - St_{t,l}/n}\left(1-\frac{St_{t,l}}{n}\right)\left(1-\frac{St_{t,l}}{n_0}\right)k_l\frac{1-e^{r_0z_t}}{r_0z_t} - \frac{e^{r_0z_t}-e^{rz_t}}{s/St_l + St_{t,l}/n_0 - St_{t,l}/n}\frac{wr_0}{St_l}\right\}(\bar{\theta}_{l,in}-\bar{\theta}_{g,in})$$

$$G_{tlg} = pG_{tll}, \quad G_{mlg} = p\left[\frac{s/St_l - St_{t,l}/n + 1}{(1-e^{rz_t})(St_{t,l}/n-1)} - 1\right]G_{mll}$$

$$G_{tgg} = p\left(\frac{s/St_l - St_{t,l}/n+1}{(1-e^{rz_t})\,e^{K}\,St_{t,l}/n} + 1\right)G_{tgl} + \frac{h_2-h_1}{h_2}\frac{2St_g\,St_f}{(s+St_g+2)^2}\frac{1}{Fo_f\,a^2} + 2St_g\,e^{-(s+St_g)}$$

$$G_{mgg} = p\left(\frac{s/St_l - St_{t,l}/n+1}{(1-e^{rz_t})\,e^{K}\,St_{t,l}/n} - 1\right)G_{mgl} + \frac{2St_g}{s+St_g+2}\frac{1}{a_0^2-a^2} \times \left\{\frac{\tanh[a_0(h_2-h_1)]}{a_0h_2} - \frac{\tanh[a(h_2-h_1)]}{ah_2}\right\}$$

$$\times \frac{St_f}{Fo_f}\left[\frac{k_g(s+2)+St_g}{s+St_g+2}\right]\frac{2}{St_g+2}\frac{D}{L}$$

$$+\frac{2St_g}{s+St_g+2}\frac{k_g-1}{St_g+2}\frac{\tanh[a_0(h_2-h_1)]}{a_0h_2}\frac{D}{L}e^{-(s+St_g)}$$

$$+\frac{h_1}{h_2}\frac{2St_{g,t}}{s+St_{g,t}+2}\frac{k_g-1}{St_{g,t}+2}\frac{D}{L}$$

$$\times e^{-(s+St_g)}\frac{1-e^{r_0z_t}}{r_0z_t}\frac{St_{t,l}}{n_0}(\bar{\theta}_{l,in}-\bar{\theta}_{g,in})$$

$$a=\sqrt{\left(\frac{1}{Fo_f}\left(s+St_f-\frac{St_fSt_g}{s+St_g+2}\right)\right)};$$

$$a_0=a(s=0);\quad n_0=n(s=0);\quad r_0=r(s=0)$$

$$r=\frac{w}{2Fo_l}-\sqrt{\left(\frac{w}{2Fo_l}\right)^2+\frac{1}{Fo_l}\left(s+St_l-\frac{St_lSt_{t,l}}{n}\right)}$$

$$n=s+St_{t,l}+St_{t,g}+Fo_t\,a\tanh[a(h_2-h_1)]-\frac{St_{t,g}St_{g,t}}{s+St_{g,t}+2}$$

$$p=\left\{\frac{2St_g}{s+St_g+2}\frac{\tanh[a(h_2-h_1)]}{ah}+\frac{h_1}{h_2}\frac{2St_{g,t}}{s+St_{g,t}+2}\right\}\frac{St_{t,l}}{n}e^K$$

$$K=\frac{\delta_t}{L}\sqrt{(s/Fo_K)};\quad Fo_K=\frac{\lambda_t t_0}{c_t\rho_t L^2};\quad w=\frac{\bar{w}_l}{\bar{w}_g}$$

tube deflection on the liquid side

$$G_u=e^{r_uz_u};\quad r_u=\frac{w}{2Fo_l}-\sqrt{\left(\frac{w^2}{4Fo_l^2}+\frac{1}{Fo_l}\left(s+St_l-\frac{St_lSt_{t,l}}{s+St_{t,l}}\right)\right)}$$

where z_u is the length of tube deflection (dimensionless). These transfer functions were slightly modified by comparison with a more sophisticated model derived in ref. [8]. The gas exit temperature is averaged by integrating along the y-coordinate.

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Approximate analytical solution to forced convection with arbitrary surface heat flux

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INTRODUCTION

AMBROK [1] has developed an approximate analytical solution for the local surface coefficient of heat transfer for steady flow of a fluid with arbitrary free stream velocity and density variation over a surface with an arbitrary specified wall temperature variation. This is accomplished by solving the integral form of the thermal energy equation for the local enthalpy thickness $\delta_t^*(x)$ by use of an assertion of a universal relation, between Nusselt or Stanton number and the enthalpy thickness Reynolds number, suggested by the result for an isothermal flat plate.

Although the Stanton number expression so developed is claimed to hold for laminar, transitional, or turbulent flow, Ambrok's [1] basic, major assumption is most closely obeyed in turbulent flows. Kays and Crawford [2] give Ambrok's method high grades in the prediction of turbulent heat transfer and point out that it does well except for severely accelerated flows where values of the acceleration parameter K exceed about 0.5×10^{-6}. Data in Orlando *et al.* [3] indicate that, at least for near equilibrium flows where K = constant, the local Stanton number for adverse pressure gradients is the same function of the enthalpy thickness Reynolds number as it is for a flat plate. This implies that Ambrok's method will give particularly good results in the case of adverse pressure gradients. Finally, an extension of Ambrok's result to flows with transpiration and to high speed flows is presented in Kays and Crawford [2].

In this day of computer generated finite difference solutions to forced convection heat transfer problems, the appeals of Ambrok's method are: the ease, simplicity, and economy with which it can be applied; the fact that it generally leads to an explicit, analytical expression for St_x; and the aforementioned relatively good accuracy for an extremely broad class of turbulent flow problems.

The purpose of this work is to show how Ambrok's basic ideas can be used to solve the problem of arbitrarily varying flux, q_w, at the wall rather than specified surface temperature variation. Specified surface heat flux situations arise in electric resistance heating, nuclear fuel rods, and often in experimental rigs where electric heaters or tape are employed. Approximate analytical expressions are developed for the Stanton number and wall temperature variation and are validated by comparison with experimental data.

ANALYSIS

Consider steady, thin boundary layer, low speed flow of a fluid with constant free stream temperature T_s over an arbitrarily shaped rotationally symmetric body. For these conditions, the integral form of the thermal energy equation can be written as [2]

$$\frac{d\delta_t^*}{dx}+\left[\frac{1}{\rho_s}\frac{d\rho_s}{dx}+\frac{1}{u_s}\frac{du_s}{dx}+\frac{1}{R}\frac{dR}{dx}+\frac{1}{(T_w-T_s)}\frac{d}{dx}(T_w-T_s)\right]\delta_t^*=\frac{q_w}{\rho_su_sC_{ps}(T_w-T_s)}. \quad (1)$$

For a specified flux $q_w(x)$ one wishes to predict $T_w(x)$ and St_x. Hence, equation (1) is rearranged, a new dependent variable $\psi=\delta_t^*(T_w-T_s)$ is defined and $B(x)$ is used to denote the first three terms in the brackets of equation (1), giving

$$\frac{d\psi}{dx}+B(x)\psi=\frac{q_w(x)}{\rho_su_sC_{ps}}. \quad (2)$$

Solving equation (2) subject to the initial condition that $\psi=\psi_0$ at $x=x_0$ yields